

ON THE PRIMITIVITY OF BIRATIONAL TRANSFORMATIONS OF IRREDUCIBLE SYMPLECTIC MANIFOLDS

FEDERICO LO BIANCO

ABSTRACT. Let $f: X \dashrightarrow X$ be a birational transformation of a complex irreducible symplectic manifold X . Some important dynamical properties of f are encoded by the induced linear automorphism f^* of $H^2(X, \mathbb{Z})$. Our main result is that a birational transformation such that f^* has at least one eigenvalue with modulus > 1 doesn't admit any invariant fibration (in particular its generic orbit is Zariski-dense).

1. INTRODUCTION

A complex manifold is said **irreducible symplectic** if it is simply connected and the vector space of holomorphic 2-forms is spanned by a nowhere degenerate form. Irreducible symplectic manifolds form, together with Calabi-Yau manifolds and complex tori, one of the three fundamental classes of Kähler manifolds with trivial canonical bundle. We are going to denote by X an irreducible symplectic manifold and by $f: X \dashrightarrow X$ a birational transformation of X .

On the second cohomology of X we can define a quadratic form, the Beauville-Bogomolov form, whose restriction to $H^{1,1}(X, \mathbb{R})$ is hyperbolic (i.e. has signature $(1, h^{1,1}(X) - 1)$) and which is preserved by the linear pull-back action f^* induced by f on cohomology; the setting is therefore similar to that of a compact complex surface, where the intersection form makes the second cohomology group into a hyperbolic lattice. In the surface case, the action of an automorphism $f: S \rightarrow S$ on cohomology translates into dynamical properties of f (see Paragraphs 3.3 and 3.4 for details), and we can hope to have similar results in the irreducible symplectic case.

If $g: M \dashrightarrow M$ is a meromorphic transformation of a compact Kähler manifold M , for $p = 0, 1, \dots, \dim(M)$ the p -th **dynamical degree** of g is

$$\lambda_p(g) := \limsup_{n \rightarrow +\infty} \|(g^n)_p^*\|^{\frac{1}{n}},$$

where $(g^n)_p^*: H^{p,p}(M) \rightarrow H^{p,p}(M)$ is the linear morphism induced by g^n and $\|\cdot\|$ is any norm on the space $\text{End}(H^{p,p}(M))$. Note that in the case of an automorphism, $\lambda_p(f)$ is just the maximal modulus of eigenvalues of f_p^* .

Let $g: M \dashrightarrow M$ be a birational transformation of a compact Kähler manifold. A meromorphic fibration $\pi: M \dashrightarrow B$ onto a compact Kähler manifold B such that $\dim B \neq 0$, $\dim X$ is called **g -invariant** if there exists a birational transformation $h: B \dashrightarrow B$ such that $\pi \circ g = h \circ \pi$.

$$\begin{array}{ccc}
M & \xrightarrow{g} & M \\
| & & | \\
\pi & & \pi \\
\downarrow & & \downarrow \\
B & \xrightarrow{h} & B
\end{array}$$

The transformation g is said to be **primitive** (see [22]) if it admits no invariant fibration.

In the surface case, an automorphism whose action on cohomology has infinite order admits an invariant fibration onto a curve if and only if all the dynamical degrees are equal to 1 (Theorem 3.8). Our main result establish an analogue of the "only if" direction.

Main Theorem. *Let X be an irreducible symplectic manifold, $f: X \dashrightarrow X$ a bimeromorphic transformation with at least one dynamical degree > 1 . Then*

- (1) *f is primitive;*
- (2) *f admits at most $\dim(X) + b_2(X) - 2$ periodic hypersurfaces;*
- (3) *the generic orbit of f is Zariski-dense.*

Here a hypersurface $H \subset X$ is said to be f -periodic if its strict transform $(f^n)^*H$ by some iterate of f is equal to H .

Remark 1.1. Point (2) follows from point (1) and [5, Theorem B]; point (3) follows from point (1) and [1, Theorem 4.1], but is proven here as a lemma (Lemma 4.6).

In order to prove the Main Theorem, we establish a result on the dynamics of birational transformations of projective manifolds that has its own interest.

Proposition. *Let X, B be projective manifolds, $f: X \dashrightarrow X, g: B \dashrightarrow B$ birational transformations and $\pi: X \rightarrow B$ a non-trivial fibration such that $\pi \circ f = g \circ \pi$. If the generic orbit of g is Zariski-dense and the generic fibre of π is of general type, then*

- (1) *π is isotrivial over an open dense subset $U \subset B$;*
- (2) *there exists an étale cover $U' \rightarrow U$ such that the induced fibration $X' = U' \times_U \pi^{-1}(U)$ is trivial: $X' \cong U' \times F$ for a fibre F ;*
- (3) *the images by the natural morphism $X' \rightarrow \pi^{-1}(U)$ of the varieties $U' \times \{pt\}$ are f -periodic; in particular the generic orbit of f is not Zariski-dense.*

Remark 1.2. Point (1) is equivalent to point (2) by [28, Proposition 2.6.10].

In Section 2 we recall the definition and main results about dynamical degrees, in the absolute and relative context; Section 3 is consecrated to irreducible symplectic manifolds, with a focus on the invariance of the Beauville-Bogomolov form under the action of a birational transformation; in Section 4 and 5 we prove the Main Theorem and the Proposition above; Section 6 presents a different approach to the proof of the Main Theorem, which allows to prove a slightly weaker version of it.

1.1. Acknowledgements. I would like to thank Serge Cantat for proposing the topic and strategy of this paper, and for his precious help in all the stages of its redaction; I am also grateful to Mathieu Romagny for the fruitful conversations about Hilbert schemes.

2. DYNAMICAL DEGREES

Throughout this section M will be a compact Kähler manifold of dimension d .

2.1. Definition and entropy. Let $f: X \dashrightarrow Y$ be a dominant meromorphic map between compact Kähler manifolds; the map f is then holomorphic outside its indeterminacy locus $\mathcal{I} \subset X$, which has codimension at least 2. The closure Γ of its graph over $X \setminus \mathcal{I}$ is an irreducible analytic subset of dimension d in $X \times Y$. Let π_X, π_Y denote the restrictions to Γ of the projections from $X \times Y$ to X and to Y respectively; then π_X induces a biholomorphism $\pi_X^{-1}(X \setminus \mathcal{I}) \cong X \setminus \mathcal{I}$ and we can identify f with $\pi_Y \circ \pi_X^{-1}$. Let α be a smooth (p, q) -form on Y ; we define the pull-back of α by f as the (p, q) -current (see [8] for the basic theory of currents) on X

$$f^* \alpha := (\pi_X)_* (\pi_Y^* \alpha).$$

It is not difficult to see that if α is closed (resp. positive), then so is $f^* \alpha$, so that f induces a linear morphism between the Hodge cohomology groups. This definition of pull-back coincides with the usual one when f is holomorphic.

Remember that the p -th dynamical degree of a dominant meromorphic map $f: M \dashrightarrow M$ are defined as

$$\lambda_p(f) = \limsup_{n \rightarrow +\infty} \|(f^n)_p^*\|^{\frac{1}{n}}.$$

Thanks to the above definition of pull-back, one can prove that

$$\lambda_p(f) = \lim_{n \rightarrow +\infty} \left(\int_M (f^n)^* \omega^p \wedge \omega^{d-p} \right)^{\frac{1}{n}}$$

for any Kähler form ω . See [12], [7] for details.

The p -th dynamical degree measures the exponential growth of the volume of $f^n(V)$ for subvarieties $V \subset M$ of dimension p [20].

Remark 2.1. By definition $\lambda_0(f) = 1$; $\lambda_d(f)$ coincides with the topological degree of f : it is equal to the number of points in a generic fibre of f .

Remark 2.2. Let f be an automorphism. Then we have $(f^n)^* = (f^*)^n$, so that $\lambda_p(f)$ is the maximal modulus of eigenvalues of the linear automorphism $f_p^*: H^{p,p}(M, \mathbb{R}) \rightarrow H^{p,p}(M, \mathbb{R})$; since f^* also preserves the positive cone $\mathcal{K}_p \subset H^{p,p}(M, \mathbb{R})$, a theorem of Birkhoff [2] implies that $\lambda_p(f)$ is a positive real eigenvalue of f_p^* . It should be noted however that in the bimeromorphic setting we have in general $(f^n)^* \neq (f^*)^n$.

Remark 2.3. If f is bimeromorphic we have

$$\lambda_p(f) = \lambda_{d-p}(f^{-1}).$$

Indeed, for f biregular we have

$$\int_M (f^n)^* \omega^p \wedge \omega^{d-p} = \int_M (f^{-n})^* (f^n)^* \omega^p \wedge (f^{-n})^* \omega^{d-p} = \int_M \omega^p \wedge (f^{-n})^* \omega^{d-p},$$

which proves the equality by taking the limit.

If f is only bimeromorphic, for all n we can find two dense open subsets $U_n, V_n \subset M$ such that f^n induces an isomorphism $U_n \cong V_n$; by the definition of pull-back the measures $(f^n)^* \omega^p \wedge \omega^{d-p}$ and $\omega^p \wedge (f^{-n})^* \omega^{d-p}$ have no mass on any proper closed analytic subset, so that

$$\int_M (f^n)^* \omega^p \wedge \omega^{d-p} = \int_{U_n} (f^n)^* \omega^p \wedge \omega^{d-p} = \int_{V_n} \omega^p \wedge (f^{-n})^* \omega^{d-p} = \int_M \omega^p \wedge (f^{-n})^* \omega^{d-p},$$

which proves the equality in the bimeromorphic case as well.

The main interest in the definition of dynamical degrees lies in the following theorem by Yomdin and Gromov [16].

Theorem 2.4. *If $f: M \rightarrow M$ is an automorphism, then the topological entropy of f is given by*

$$h_{top}(f) = \max_{p=0,\dots,d} \log \lambda_p(f).$$

The topological entropy is a positive real number which measures the disorder created by iterations of f .

It is also possible to give a definition of topological entropy in the bimeromorphic context (see [13]), but in this situation we only have

$$h_{top}(f) \leq \max_{p=0,\dots,d} \log \lambda_p(f).$$

2.2. Relative setting. Dinh, Nguyễn and Truong have studied the behaviour of dynamical degrees in the relative setting ([10] and [11]). Throughout this paragraph we denote by $f: M \dashrightarrow M$ a meromorphic transformation of a compact Kähler manifold M of dimension d , by $\pi: M \dashrightarrow B$ a meromorphic fibration onto a compact Kähler manifold B of dimension k and by $g: B \dashrightarrow B$ a meromorphic transformation such that

$$g \circ \pi = \pi \circ f.$$

The p -th **relative dynamical degree** of f is defined as

$$\lambda_p(f|\pi) = \limsup_{n \rightarrow +\infty} \left(\int_M (f^n)^* \omega_M^p \wedge \pi^* \omega_B^k \wedge \omega_M^{d-p-k} \right)^{\frac{1}{n}},$$

where ω_M and ω_B are arbitrary Kähler forms on M and B respectively. In particular $\lambda_p(f|\pi) = 0$ for $p > d - k$.

Roughly speaking, $\lambda_p(f|\pi)$ gives the exponential growth of $(f^n)^*$ acting on the subspace of classes in $H^{p+k,p+k}(M, \mathbb{R})$ that can be supported on a generic fibre of π ; if M is projective, it also represents the growth of the volume of $f^n(V)$ for p -dimensional subvarieties $V \subset \pi^{-1}(b)$ of a generic fibre of π .

Remark 2.5. Dynamical degrees and relative dynamical degrees are bimeromorphic invariants [10]. In other words, if there exist bimeromorphic maps $\phi: M \dashrightarrow M'$, $\psi: B \dashrightarrow B'$ and a meromorphic fibration $\pi': M' \dashrightarrow B'$ such that $\pi' \circ \phi = \psi \circ \pi$, then

$$\lambda_p(f) = \lambda_p(\phi \circ f \circ \phi^{-1}), \quad \lambda_q(f|\pi) = \lambda_q(\phi \circ f \circ \phi^{-1}|\pi').$$

Remark 2.6. If $F = g^{-1}(b)$ is a regular, f -invariant, non-multiple fibre, then $\lambda_p(f|\pi) = \lambda_p(f|_F)$ for all p (see [10]).

The following theorem is due to Dinh, Nguyễn and Truong [10].

Theorem 2.7. *Let M be a compact Kähler manifold, $f: M \dashrightarrow M$ a meromorphic transformation, $\pi: M \dashrightarrow B$ a meromorphic fibration and $g: B \dashrightarrow B$ a meromorphic transformation such that $\pi \circ f = g \circ \pi$. Then for all $p = 0, \dots, \dim(M)$*

$$\lambda_p(f) = \max_{q+r=p} \lambda_q(f|\pi) \lambda_r(g).$$

2.3. Log-concavity. Dynamical degrees and their relative counterparts enjoy a log-concavity property (see [23],[29], [16], [7] for the original result, [10] for the relative setting).

Proposition 2.8. *If $f: M \dashrightarrow M$ is a meromorphic dominant map, the sequence $p \mapsto \log \lambda_p(f)$ is concave on the set $\{0, 1, \dots, d\}$; in other words*

$$\lambda_p(f)^2 \geq \lambda_{p-1}(f)\lambda_{p+1}(f) \quad \text{for } p = 1, \dots, d-1.$$

Analogously, if $\pi: M \dashrightarrow B$ is an f -invariant meromorphic fibration, then the sequence $p \mapsto \log \lambda_p(f|\pi)$ is concave on the set $\{0, 1, \dots, \dim(M) - \dim(B)\}$.

As a consequence we have $\lambda_p \geq 1$ for all $p = 0, \dots, d$; furthermore, there exist $0 \leq p \leq p+q \leq d$ such that

$$(2.1) \quad 1 = \lambda_0(f) < \dots < \lambda_p(f) = \lambda_{p+1}(f) = \dots = \lambda_{p+q}(f) > \dots > \lambda_d(f).$$

3. IRREDUCIBLE SYMPLECTIC MANIFOLDS

We give here the basic notions and properties of irreducible symplectic manifolds (see [17], [25] for details).

Remark 3.1. Because of the non-degeneracy of σ , one can easily prove that an irreducible symplectic manifold has even complex dimension.

Throughout this section X denotes an irreducible symplectic manifold of dimension $2n$ and σ a non-degenerate holomorphic two-form on X .

Here is a list of the known examples of such manifolds that are not deformation equivalent.

- (1) Let S be a $K3$ surface, i.e. a simply connected Kähler surface with trivial canonical bundle. Then the Hilbert scheme $S^{[n]} = \text{Hilb}^n(S)$, parametrizing 0-dimensional subschemes of S of length n , is a $2n$ -dimensional irreducible symplectic manifold.
- (2) Let T be a complex torus of dimension 2, let $\phi: \text{Hilb}^n(T) \rightarrow \text{Sym}^n(T)$ be the natural morphism and let $s: \text{Sym}^n(T) \rightarrow T$ be the sum morphism. Then the kernel $K_{n-1}(T)$ of the composition $s \circ \phi$ is an irreducible symplectic manifold of dimension $2n - 2$, which is called a *generalized Kummer variety*.
- (3) O'Grady has found two sporadic examples of irreducible symplectic manifolds of dimension 6 and 10.

An irreducible symplectic manifold is said of type $K3^{[n]}$ (respectively of type generalized Kummer) if it is deformation equivalent to $\text{Hilb}^n(S)$ for some $K3$ surface S (respectively to $K_{n-1}(T)$ for some two-dimensional complex torus T).

3.1. The Beauville-Bogomolov form. We can define a natural quadratic form on the second cohomology $H^2(X, \mathbb{R})$ which enjoys similar properties to the intersection form on compact surfaces; for details and proofs see [17].

Definition 3.2. *Let σ be a holomorphic two-form such that $\int (\sigma \bar{\sigma})^n = 1$. The Beauville-Bogomolov quadratic form q_{BB} on $H^2(X, \mathbb{R})$ is defined by*

$$q_{BB}(\alpha) = \frac{n}{2} \int_X \alpha^2 (\sigma \bar{\sigma})^{n-1} + (1-n) \left(\int_X \alpha \sigma^n \bar{\sigma}^{n-1} \right) \left(\int_X \alpha \sigma^{n-1} \bar{\sigma}^n \right).$$

The Beauville-Bogomolov form satisfies two important properties: first the Beauville-Fujiki relation, saying that there exists a constant $c > 0$ such that

$$q_{BB}(\alpha)^n = c \int_X \alpha^{2n} \quad \text{for all } \alpha \in H^2(X, \mathbb{R}).$$

In particular, some multiple of q_{BB} is defined over \mathbb{Z} .

Second, the next Proposition describes completely the signature of the form.

Proposition 3.3. *The Beauville-Bogomolov form has signature $(3, b_2(X) - 3)$ on $H^2(X, \mathbb{R})$. More precisely, the decomposition $H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R}) \oplus (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ is orthogonal with respect to q_{BB} , and q_{BB} has signature $(1, h^{1,1}(X) - 1)$ on $H^{1,1}(X, \mathbb{R})$ and is positive definite on $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$.*

Remark 3.4. For a divisor $D \in \text{Div}(X)$, we define $q_{BB}(D) := q_{BB}(c_1(\mathcal{O}_X(D)))$.

If D is effective and without fixed components, then $q_{BB}(D) \geq 0$. Indeed, let D' be an effective divisor linearly equivalent to D and with no components in common with D . We have

$$q_{BB}(D) = \frac{n}{2} \int_{D \cap D'} (\sigma \bar{\sigma})^{n-1},$$

where each irreducible component of the intersection $D \cap D'$ is counted with its multiplicity. The integral on the right hand side is non-negative because σ is a holomorphic form.

If furthermore D is ample, then by Beauville-Fujiki relation $q_{BB}(D) > 0$.

3.2. Bimeromorphic maps between irreducible symplectic manifolds. A bimeromorphic map $f: M \dashrightarrow M'$ between compact complex manifolds is an isomorphism in codimension 1 if there exist dense open subsets $U \subset M$ and $U' \subset M'$ such that

- (1) $\text{codim}(X \setminus U) \geq 2, \text{codim}(X' \setminus U') \geq 2$;
- (2) f induces an isomorphism $U \cong U'$.

A **pseudo-automorphism** of a complex manifold X is a bimeromorphic transformation which is an isomorphism in codimension 1.

Proposition 3.5 (Proposition 21.6 and 25.14 in [17]). *Let $f: X \dashrightarrow X'$ be a bimeromorphic map between irreducible symplectic manifolds. Then f is an isomorphism in codimension 1 and induces a linear isomorphism $f^*: H^2(X', \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$ which preserves the Beauville-Bogomolov form.*

In particular, the group of birational transformation of an irreducible symplectic manifold X coincides with its group of pseudo-automorphisms and acts by isometries on $H^2(X, \mathbb{Z})$.

3.3. Isometries of hyperbolic spaces. Proposition 3.3 establishes a parallel between the dynamics of automorphisms of compact Kähler surfaces and that of bimeromorphic transformations of irreducible symplectic manifolds: in both cases the map on the manifold induces an isomorphism at the level of the integral cohomology group $H^2(X, \mathbb{Z})$ preserving a non-degenerate quadratic form (the intersection form in the surface case and the Beauville-Bogomolov form in the irreducible symplectic one). By Hodge's index theorem, the intersection form on the Picard group of a surface S has signature $(1, \rho(S) - 1)$, which leads to a classification of automorphisms of surfaces as loxodromic, parabolic or elliptic depending on their action on the hyperbolic lattice $NS_{\mathbb{Z}}(S)$ (see [6]).

Analogously if X is an irreducible symplectic manifold, the restriction of the Beauville-Bogomolov form to $H^{1,1}(X, \mathbb{R})$ has signature $(1, h^{1,1}(X) - 1)$. Since $H^{1,1}(X, \mathbb{R})$ is invariant by the action of a bimeromorphic transformation $f: X \dashrightarrow X$, and since the two lines $\mathbb{C}\sigma$ and $\mathbb{C}\bar{\sigma}$ are also invariant (the action of f^* being given by multiplication by a complex number of modulus 1), we can also classify bimeromorphic transformations of irreducible symplectic manifolds depending on their action on $H^2(X, \mathbb{Z})$ as follows.

Definition 3.6. *Let $f: X \dashrightarrow X$ a bimeromorphic transformation of an irreducible symplectic manifold (respectively, an automorphism of a compact Kähler surface) and denote by f_1^* the linear automorphism of $H^{1,1}(X, \mathbb{R})$ induced by f . We say that f is*

- loxodromic if f_1^* admits an eigenvalue of modulus strictly greater than 1 (or, equivalently, if $\lambda_1(f) > 1$);
- parabolic if all the eigenvalues of f_1^* have modulus 1 and $\|(f^n)_1^*\|$ is not bounded as $n \rightarrow +\infty$;
- elliptic if $\|(f^n)_1^*\|$ is bounded as $n \rightarrow +\infty$.

In each of the cases above, simple linear algebra arguments allow to further describe the situation.

Denote by $\mathcal{C}_{\geq 0} \subset H^{1,1}(X, \mathbb{R})$ (respectively $\mathcal{C}_0 \subset H^{1,1}(X, \mathbb{R})$) the positive (resp. null) cone for the Beauville-Bogomolov (respectively, intersection) form q :

$$\mathcal{C}_{\geq 0} = \{\alpha \in H^{1,1}(X, \mathbb{R}) | q(\alpha) \geq 0\},$$

$$\mathcal{C}_0 = \{\alpha \in H^{1,1}(X, \mathbb{R}) | q(\alpha) = 0\}.$$

\mathcal{C}_0 is called the *isotropic cone* for the Beauville-Bogomolov form.

For a proof of the following result, see [27] (for irreducible symplectic manifolds) and [6] (for surfaces).

Theorem 3.7. *Let $f: X \dashrightarrow X$ a bimeromorphic transformation of an irreducible symplectic manifold (respectively, an automorphism of a compact Kähler surface).*

- If f is loxodromic, then f_1^* has exactly one eigenvalue with modulus > 1 and exactly one eigenvalue with modulus < 1 ; these eigenvalues are real, simple and they are the inverse of each another; their eigenspaces are contained in \mathcal{C}_0 , they are the only f_1^* -invariant lines in $\mathcal{C}_{\geq 0}$ and they are not defined over \mathbb{Z} .
- If f is parabolic, then all eigenvalues of f_1^* are roots of unity; the Jordan form of f^* has exactly one non-trivial Jordan block, which is of dimension 3 (in other words $\|(f^n)_1^*\|$ has quadratic growth); for every $\alpha \in H^{1,1}(X, \mathbb{R})$, $(f^n)_1^* \alpha / n^2$ converges to a class contained in \mathcal{C}_0 , which (for every α outside a proper subspace of $H^{1,1}(X, \mathbb{R})$) spans the only f_1^* -invariant line of $\mathcal{C}_{\geq 0}$.
- If f is elliptic, then some iterate of f_1^* is equal to the identity.

3.4. The parabolic case. In the case of surfaces, an automorphism being of parabolic type has a clear geometric interpretation (see [14], [15], or [9] for the birational case).

Theorem 3.8. *Let S be a compact Kähler surface; an automorphism $f: S \rightarrow S$ is of parabolic type if and only if there exists an f -invariant fibration $\pi: S \rightarrow C$ onto a nonsingular compact curve C .*

We could expect the situation to be similar in the irreducible symplectic context; indeed, Hu, Keum and Zhang have proved a partial analogue to Theorem 3.8, see [22]:

Theorem 3.9. *Let X be a $2n$ -dimensional projective irreducible symplectic manifold of type $K3^{[n]}$ or of type generalized Kummer and let $f \in \text{Bir}(X)$ be a bimeromorphic transformation which is not elliptic; f is parabolic if and only if it admits a rational Lagrangian invariant fibration $\pi: X \dashrightarrow \mathbb{P}^n$ such that the induced transformation on \mathbb{P}^n is biregular, i.e. there exists $g \in \text{Aut}(\mathbb{P}^n)$ such that $\pi \circ f = g \circ \pi$.*

The hard direction is to exhibit an invariant fibration for a parabolic transformation. The Main Theorem generalizes the converse, proving that the dynamics of a loxodromic transformation is too complicated to expect an invariant fibration.

4. PROOF OF THE MAIN RESULTS

Throughout this section, $f: X \dashrightarrow X$ denotes a loxodromic bimeromorphic transformation of an irreducible symplectic manifold X , $\pi: X \dashrightarrow B$ a meromorphic invariant fibration onto a Kähler manifold B such that $0 < \dim B < \dim X$ and $g: B \dashrightarrow B$ the induced transformation of the base.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow \pi & & \downarrow \pi \\ B & \xrightarrow{g} & B \end{array}$$

The results in this Section are largely inspired by [1].

4.1. Meromorphic fibrations on irreducible symplectic manifolds. We collect here some useful facts about the fibration π .

Remark 4.1. If B is Kähler, then it is projective. Indeed, if B wasn't projective, by Kodaira's projectivity criterion and Hodge decomposition

$$H^2(B, \mathbb{C}) = H^{2,0}(B) \oplus H^{1,1}(B) \oplus H^{0,2}(B),$$

we would have $H^{2,0}(B) \neq \{0\}$, meaning that B carries a non-trivial holomorphic 2-form σ_B . Since the indeterminacy locus of π has codimension at least 2, the pull-back $\pi^* \sigma_B$ can be extended to a global non-trivial 2-form on X which is not a multiple of σ , contradicting the hypothesis on X .

Here we use the same conventions as in [1]: let $\eta: \tilde{X} \rightarrow X$ be a resolution of the indeterminacy locus of π (see [31]), and let $\nu: \tilde{X} \rightarrow B$ be the induced holomorphic fibration, whose generic fibre is bimeromorphic to that of π .

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow \eta & \searrow \nu & \\ X & \xrightarrow{\pi} & B \end{array}$$

The pull-back $\pi^* D$ of an effective divisor $D \in \text{Div}(B)$ is defined as

$$\pi^* D = \eta_* \nu^* D,$$

where η_* is the pushforward as cycles. The pull-back induces linear morphisms $\text{Pic}(B) \rightarrow \text{Pic}(X)$ and $NS(B) \rightarrow NS(X)$, and is compatible with the pull-back of smooth forms defined in Section 2.

Now let $H \in \text{Pic}(B)$ be an ample class, and let $L = \pi^* H$. The pull-back of the complete linear system $|H|$ is a linear system $U \subset |L|$, whose associated meromorphic fibration is exactly π . In particular, L has no fixed component, and by Remark 3.4 we have $q_{BB}(L) \geq 0$.

Let $NS(B) \subset H^{1,1}(B, \mathbb{R})$ denote the Neron-Severi group with real coefficients of B . The following Lemma is essentially proven in [1].

Lemma 4.2. *The restriction of the Beauville-Bogomolov form to the pull-back $\pi^* NS(B)$ is not identically zero if and only if the generic fibre of π is of general type. If this is the case, then X is projective.*

Proof. Remark first that, since the generic fibre of ν is bimeromorphic to the generic fibre of π and the Kodaira dimension is a bimeromorphic invariant, the generic fibre of π is of general type if and only if the generic fibre of ν is.

As a second remark, by [26] if there exists a big line bundle on a compact Kähler manifold X , then X is projective.

Suppose that the generic fibre of π is of general type. Let H be an ample divisor on B and let $L = \pi^*H$. By [1][Theorem 2.3] we have $\kappa(X, L) = \dim(B) + \kappa(F)$, where F is the generic fibre of ν ; we conclude that L is big (and in particular X is projective). We can thus write $L = A + E$ for an ample divisor A and an effective divisor E on X . Now, if q denotes the Beauville-Bogomolov form, we have

$$q(L) = q(L, A) + q(L, E) \geq q(L, A) = q(A, A) + q(A, E) \geq q(A, A) > 0,$$

where the first and second inequalities are consequences of L and A being without fixed components and the last one follows directly from Remark 3.4. This proves the "if" direction.

Now assume that the restriction of q_{BB} to $\pi^*NS(B)$ is not identically zero. Since ample classes generate $NS_{\mathbb{R}}(B)$, there exists an ample line bundle $H \in \text{Pic}(B)$ such that, denoting $L = \pi^*H$, $q(L) \neq 0$; furthermore, L is without fixed components, so that $q(L) > 0$ by Remark 3.4. It follows by [3][Theorem 4.3.i] that L is big (thus X is projective), and so is η^*L since η is a birational morphism. Therefore, the restriction $\eta^*L|_F$ to a generic fibre of ν is also big (see [24][Corollary 2.2.11]). Now we have

$$\eta^*L = \nu^*H + \sum a_i E_i \quad \text{for some } a_i \geq 0,$$

where the sum runs over all the irreducible components of the exceptional divisor of η . The adjunction formula leads to

$$K_F = K_{\tilde{X}}|_F + \det N_{F/\tilde{X}}^* = K_{\tilde{X}}|_F + \sum e_i E_i|_F \quad \text{for some } e_i > 0,$$

since the conormal bundle $N_{F/\tilde{X}}^*$ is trivial.

This implies that, for some $m > 0$, the divisor $mK_F - \eta^*L|_F$ is effective because $\nu^*H|_F$ is trivial. Thus

$$\kappa(F) \geq \kappa(F, \eta^*L|_F) = \dim(F),$$

meaning that F is of general type. This proves the "only if" direction. \square

Corollary 4.3. *If the generic fibre of ν is not of general type, then $\pi^*NS(B) \subset H^{1,1}(X, \mathbb{R})$ is a line contained in the isotropic cone \mathcal{C}_0 .*

Proof. By Lemma 4.2, $\pi^*NS(B)$ is contained in the isotropic cone. The pull-back L of an ample line bundle on B is effective and non-trivial, so that its numerical class is also non-trivial; thus $\pi^*NS(B)$ cannot be trivial. To conclude it suffices to remark that $\pi^*NS(B)$ is a linear subspace of $H^{1,1}(X, \mathbb{R})$, and the only non-trivial subspaces contained in the isotropic cone are lines. \square

4.2. Density of orbits. The following theorem was proven in [1].

Theorem 4.4. *Let X be a compact Kähler manifold and let $f: X \dashrightarrow X$ be a dominant meromorphic endomorphism. Then there exists a dominant meromorphic map $\pi: X \dashrightarrow B$ onto a compact Kähler manifold B such that*

- (1) $\pi \circ f = \pi$;
- (2) *the general fibre X_b of π is the Zariski closure of the orbit by f of a generic point of X_b .*

Lemma 4.5. *Let $\phi: X \dashrightarrow Y$, $\psi: Y \dashrightarrow Z$ be meromorphic maps between compact complex manifolds. If ϕ is an isomorphism in codimension 1, then for all $D \in \text{Div}(Z)$*

$$(\psi \circ \phi)^*D = \phi^*\psi^*D.$$

Proof. Let $U \subset X$, $V \subset Y$ two open sets such that ϕ induces an isomorphism $U \cong V$ and such that $\text{codim}(X \setminus U) \geq 2$, $\text{codim}(Y \setminus V) \geq 2$. It is easy to see that, for every effective divisor $D_Y \in \text{Div}(Y)$, we have an equality $\phi^* D_Y = \overline{\phi|_U^*(D_Y \cap V)}$; therefore the equality is true for every divisor in $\text{Div}(Y)$.

Up to shrinking V to some other open subset whose complement has codimension at least 2, we can suppose that ψ is regular on V ; therefore the composition $\psi \circ \phi$ is regular on U , and since the complement of U has codimension ≥ 2 in X , for all $D \in \text{Div}(Z)$ we have

$$(\psi \circ \phi)^* D = \overline{(\psi \circ \phi)|_U^* D} = \overline{\phi|_U^*(\psi|_V^* D)} = \overline{\phi|_U^*(\psi^* D \cap V)} = \phi^* \psi^* D,$$

where the third equality follows again from the fact that the complement of V has codimension at least 2 in Y . This proves the claim. \square

Let us prove point (3) of the Main Theorem.

Lemma 4.6. *Let $f: X \dashrightarrow X$ be a bimeromorphic loxodromic transformation of an irreducible symplectic manifold. Then the generic orbit of f is Zariski-dense.*

Proof. If the claim were false, then by Theorem 4.4 we could construct a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow \pi & & \downarrow \pi \\ B & \xrightarrow{id} & B \end{array}$$

where π is a meromorphic map whose general fibre X_b coincides with the Zariski-closure of the f -orbit of a generic point of X_b . Remark 4.1 applies in the case where the fibres are not connected; therefore the base B is projective.

Now, remark that f^* acts as the identity on the space $\pi^* NS(B) \subset NS(X)$, which is defined on \mathbb{Q} : indeed, for $v \in NS(B)$, we have

$$f^* \pi^* v = (\pi \circ f)^* v = (id_B \circ \pi)^* v = \pi^* v,$$

where the first equality follows from Lemma 4.5; by Theorem 3.7, the Beauville-Bogomolov form is negative definite on $\pi^* NS(B)$.

Now, let $H \in \text{Pic}(B)$ be an ample line bundle and let $L = \pi^* H$. We have seen in 4.1 (again the hypothesis on fibres being connected was irrelevant) that L is a numerically non-trivial line bundle such that $q_{BB}(L) \geq 0$, contradiction. This proves the claim. \square

4.3. The key lemma. The following key lemma, together with the Proposition in Section 1, implies the Main Theorem.

Lemma 4.7 (Key lemma). *Let X be an irreducible symplectic manifold, $f: X \dashrightarrow X$ a loxodromic bimeromorphic transformation and $\pi: X \dashrightarrow B$ a meromorphic f -invariant fibration onto a compact Kähler manifold. Then X is projective and the generic fibre of π is of general type.*

Proof. Let $g: B \dashrightarrow B$ be a bimeromorphic transformation such that $g \circ \pi = \pi \circ f$.

Let us define

$$V := \text{Span} \{ (h \circ \pi)^* NS_{\mathbb{R}}(B) | h: B \dashrightarrow B \text{ birational transformation} \} \subset NS_{\mathbb{R}}(X).$$

The linear subspace V is clearly defined over \mathbb{Q} . Since the pull-back by π of an ample class is numerically non-trivial, we also have $V \neq \{0\}$.

Furthermore, V is f^* -invariant: if $v = (h \circ \pi)^* w$ for some $w \in NS(B)$ and for some birational transformation $h: B \dashrightarrow B$, then

$$f^* v = f^* (h \circ \pi)^* w = (h \circ \pi \circ f)^* w = (h \circ g \circ \pi)^* w = (\tilde{h} \circ \pi)^* w,$$

where $\tilde{h} = h \circ g: B \dashrightarrow B$ is a birational transformation and the second equality follows from Lemma 4.5.

Now suppose that the generic fibre of π is not of general type; we are first going to show that V is contained in the isotropic cone $\mathcal{C}_0 = \{v \in H^{1,1}(X, \mathbb{R}) | q_{BB}(v) = 0\}$. The generic fibre of the meromorphic fibration $h \circ \pi$ is bimeromorphic to that of π . By Lemma 4.2 we know that $(h \circ \pi)^* NS_{\mathbb{R}}(B)$ is contained in the isotropic cone for all birational transformations $h: B \dashrightarrow B$. We just need to show that for all birational transformations of B onto itself h_i, h_j and for all $w_i, w_j \in NS_{\mathbb{R}}(B)$ we have

$$q_{BB}((h_i \circ \pi)^* w_i, (h_j \circ \pi)^* w_j) = 0.$$

Let $h = h_j \circ h_i^{-1}$, and let $\rho: \tilde{B} \rightarrow B$ be a resolution of the indeterminacy locus of h ; denote by $\tilde{h}: \tilde{B} \rightarrow B$ the induced holomorphic transformation, and let $\tilde{\pi} = \rho^{-1} \circ h_i \circ \pi: \tilde{X} \dashrightarrow \tilde{B}$; $\tilde{\pi}$ is a meromorphic fibration onto the birational model \tilde{B} , whose generic fibre is bimeromorphic to that of g . Finally, let $\eta: \tilde{X} \rightarrow X$ be a resolution of singularities of $\tilde{\pi}$ and let $\nu: \tilde{X} \rightarrow \tilde{B}$ be the induced holomorphic map.

$$\begin{array}{ccccc} \tilde{X} & & & & \\ \eta \downarrow & \searrow \nu & & & \\ & & \tilde{B} & & \\ & \nearrow \tilde{\pi} & \downarrow \rho & \searrow \tilde{h} & \\ X & \xrightarrow{h_i \circ \pi} & B & \xrightarrow{h} & B \end{array}$$

Now it is clear that $\eta: \tilde{X} \rightarrow X$ is a resolution of singularities of both $h_i \circ \pi$ and $h_j \circ \pi = h \circ h_i \circ \pi$. Therefore

$$(h_i \circ \pi)^* w_i = \eta_* \nu^* \rho^* w_i = \tilde{\pi}^* \rho^* w_i \in \tilde{\pi}^* NS(\tilde{B})$$

and

$$(h_j \circ \pi)^* w_j = \eta_* \nu^* \tilde{h}^* w_j = \tilde{\pi}^* \tilde{h}^* w_j \in \tilde{\pi}^* NS(\tilde{B}).$$

Since the fibres of $\tilde{\pi}$ are not of general type, it suffices to apply Lemma 4.2 to the fibration $\tilde{\pi}: \tilde{X} \dashrightarrow \tilde{B}$ to conclude that $q_{BB}((h_i \circ \pi)^* w_i, (h_j \circ \pi)^* w_j) = 0$. This proves that V is contained in the isotropic cone.

Now the only non trivial vector subspaces of $NS_{\mathbb{R}}(X)$ contained in the isotropic cone are lines; by Theorem 3.7, V is then an f^* -invariant line contained in the isotropic cone and not defined over \mathbb{Q} . But this contradicts the definition of V . We have thus proved that the generic fibre of π is of general type.

In order to prove that X is projective it suffices to apply the last part of Lemma 4.2. \square

By [31, Corollary 14.3]) we know that the group of birational transformations of a variety of general type is finite. Therefore, we expect the dynamics of f on the fibres to be simple.

4.4. Relative Iitaka fibration. Before giving the proof of the Main Theorem, we are going to recall the basic results about the relative Iitaka fibration. We will follow the approach of [30] with some elements from [31]. See also [19], [18].

Let X be a smooth projective variety, and suppose that some multiple of K_X has some non trivial section. Recall that, for $m > 0$ divisible enough, the rational map

$$\begin{aligned} \phi_{|mK_X|}: X &\dashrightarrow \mathbb{P}H^0(X, mK_X)^\vee \\ p &\mapsto \{s \in \mathbb{P}H^0(X, mK_X) | s(p) = 0\} \end{aligned}$$

has connected fibres. Moreover the rational map $\phi_{|mK_X|}$ eventually stabilize to a rational fibration that we call *canonical fibration* of X .

Remark 4.8. If $f: X \dashrightarrow X$ is a bimeromorphic transformation of X , the pull-back of forms induces a linear automorphism $f^*: H^0(X, mK_X) \rightarrow H^0(X, mK_X)$. For example, for $m = 1$ a section $\sigma \in H^0(X, K_X)$ is a holomorphic d -form ($d = \dim X$); f is defined on an open set $U \subset X$ such that $X \setminus U$ has codimension at least 2. Therefore by Hartogs theorem the pull-back $f|_U^* \sigma$ can be extended to X . It is easy to see that the construction is invertible and induces a linear automorphism of $\mathbb{P}H^0(X, mK_X)^\vee$ which commutes with the Iitaka fibration:

$$\begin{array}{ccc} X & \dashrightarrow & X \\ \downarrow \phi_{|mK_X|} & & \downarrow \phi_{|mK_X|} \\ \mathbb{P}H^0(X, mK_X)^\vee & \xrightarrow{\tilde{f}} & \mathbb{P}H^0(X, mK_X)^\vee \end{array}$$

The above construction can be generalized to the relative setting: let $\pi: X \rightarrow B$ be a regular fibration onto a smooth projective variety B , and let $K_{X/B} = K_X \otimes \pi^* K_B^{-1}$ be the relative canonical bundle.

For some fixed positive integer $m > 0$ (divisible enough), let $\mathcal{S} = \pi_*(mK_{X/B})^\vee$. \mathcal{S} is a coherent sheaf over B ; therefore one can construct (generalizing the construction of the projective bundle associated to a vector bundle, see [31] for details) the algebraic projective fibre space

$$\eta: \text{Proj}(\mathcal{S}) \rightarrow B$$

associated to \mathcal{S} , which is a projective scheme (a priori neither reduced nor irreducible) Y over B . Its generic geometric fibre Y_b over a generic point $b \in B$ is canonically isomorphic to $\mathbb{P}H^0(X_b, mK_{X_b})^\vee$. The Iitaka morphisms $\phi_b: X_b \dashrightarrow \mathbb{P}H^0(X_b, mK_{X_b})^\vee$ induce a rational map $\phi: X \dashrightarrow Y$ over B .

The *relative canonical fibration* of X with respect to π is

$$\begin{aligned} \phi: X &\dashrightarrow Y \\ x \in X_b &\mapsto [\{s \in H^0(X_b; mK_{X_b}) | s(x) = 0\}] \in Y_b. \end{aligned}$$

It can be shown that, for m divisible enough:

- ϕ stabilizes to a certain rational fibration;
- the image by ϕ of the generic fibre $X_b = \pi^{-1}(b)$ of π is contained inside the fibre $\eta^{-1}(b)$ of the natural projection $\eta: Y \rightarrow B$;
- the restriction of ϕ to a generic fibre X_b is birationally equivalent to the canonical fibration of X_b .

Remark 4.9. The construction in Remark 4.8 can also be generalized to the relative setting: let $f: X \dashrightarrow X$ and $g: B \dashrightarrow B$ be birational transformations such that $\pi \circ f = g \circ \pi$.

For a generic $b \in B$ define

$$\begin{aligned} \tilde{f}|_{Y_b}: \mathbb{P}H^0(X_b, mK_{X_b})^\vee &\dashrightarrow \mathbb{P}H^0(X_{g(b)}, mK_{X_{g(b)}})^\vee \\ [s^*] &\mapsto \{[s] \in \mathbb{P}H^0(X_{g(b)}, mK_{X_{g(b)}}) \mid s^*(f^*s) = 0\}. \end{aligned}$$

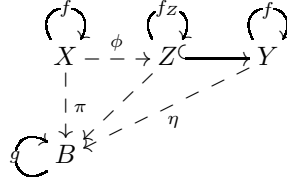
These are well defined linear automorphisms because, for a fibre X_b of π not contained in the indeterminacy locus of f , the restriction $f: X_b \dashrightarrow X_{g(b)}$ is a birational map, and thus induces a linear isomorphism

$$f^*: H^0(X_{g(b)}, mK_{X_{g(b)}}) \rightarrow H^0(X_b, mK_{X_b}).$$

Furthermore the \tilde{f}_{X_b} can be glued to a birational transformation $\tilde{f}: Y \dashrightarrow Y$ such that $\eta \circ \tilde{f} = g \circ \eta$.

Now suppose the generic fibre of π is of general type. Since the restriction of ϕ to a generic fibre of g is birational onto its image and the images of fibres are disjoint, ϕ itself must be birational onto its image; denote by Z the closure of the image of ϕ and let $f_Z = \phi \circ f \circ \phi^{-1}: Z \dashrightarrow Z$ be the birational transformation induced by f .

By the above Remark, f_Z is the restriction of the birational transformation $\tilde{f}: Y \dashrightarrow Y$. In particular f_Z induces an isomorphism between generic fibres of $\eta|_Z$.



4.5. Proof of the Main Theorem.

Lemma 4.10. *Let X, B be projective manifolds, $f: X \dashrightarrow X$ and $g: B \dashrightarrow B$ birational transformations and $\pi: X \rightarrow B$ a fibration such that $\pi \circ f = g \circ \pi$.*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi \downarrow & & \downarrow \pi \\ B & \xrightarrow{g} & B \end{array}$$

Assume that the generic fibre of π is of general type and that the generic orbit of g is Zariski-dense. Then all the fibres over a non-empty Zariski open subset of B are isomorphic.

Proof. Denote as before

$$\phi: X \dashrightarrow Y$$

the relative Iitaka fibration. We are going to identify X with its birational model $\overline{\phi(X)}$.

Let $F = \pi^{-1}(b_0)$ be the fibre of π over a point b_0 whose orbit is Zariski-dense in B , and let

$$\mathfrak{I} := \mathfrak{I}so_{mB}(X, F \times B)$$

be the B -scheme of isomorphisms over B between X and $F \times B$; the fibre \mathfrak{I}_b parametrizes the isomorphisms $X_b \cong F$. We can realize \mathfrak{I} as an open subset of the Hilbert scheme $\mathfrak{H}ilb_B(X \times_B (B \times F))$ by identifying a morphism $X_b \rightarrow F$ with its graph in $X_b \times F$. Therefore,

$$\mathfrak{I} = \coprod_{P \in \mathbb{Q}[\lambda]} \mathfrak{I}^P,$$

where the fibre \mathfrak{I}_b^P is the (a priori non irreducible and non reduced) quasi-projective scheme of (graphs of) isomorphisms $X_b \xrightarrow{\sim} F$ having fixed Hilbert polynomial $P(\lambda)$; such polynomials are calculated with respect to the restriction to the fibre $X_b \times F$ of a fixed line bundle L on $X \times_B (B \times F)$ relatively very ample over B . We shall fix

$$L = H_Y|_X \boxtimes_B H_F,$$

where H_Y is a very ample line bundle on Y and H_F is a very ample line bundle on F . Now, the pull-back of forms by f induces a linear isomorphism

$$\tilde{f}_b: \mathbb{P}H^0(X_b, mK_{X_b})^\vee \xrightarrow{\sim} \mathbb{P}H^0(X_{g(b)}, mK_{X_{g(b)}})^\vee$$

between fibres of $\eta: Y \xrightarrow{\sim} B$, which restricts to an isomorphism $X_b \rightarrow X_{g(b)}$; under the canonical identification of fibres of η with \mathbb{P}^N , $H_Y|_{Y_b} \cong \mathcal{O}_{\mathbb{P}^N}(d)$ (meaning that the section $H_Y|_{Y_b}$ has degree d) for some $d > 0$ independent of the fibre. Under the identification, the action of \tilde{f}_b is linear, so that $\tilde{f}_b^*(H_Y|_{Y_{g(b)}})$ also has degree d on \mathbb{P}^N . In particular we have

$$\tilde{f}_b^*(H_Y|_{X_{g(b)}}) = H_Y|_{X_b}.$$

Now take any isomorphism $X_{b_0} \xrightarrow{\sim} F$, which we can identify with its graph $\Gamma \subset X_{b_0} \times F$; the image of Γ by the isomorphism $\tilde{f}_{b_0} \times id_F: X_{b_0} \times F \xrightarrow{\sim} X_{g(b_0)} \times F$ is the graph Γ' of an isomorphism $X_{g(b_0)} \xrightarrow{\sim} F$. Furthermore, since $(\tilde{f}_{b_0} \times id_F)^*(L|_{X_{g(b_0)} \times F}) = L|_{X_{b_0} \times F}$, Γ' has the same Hilbert polynomial as Γ . Iterating this reasoning we find that for some $P \in \mathbb{Q}[\lambda]$ the image of the natural morphism $\psi: \mathfrak{I}^P \rightarrow B$ is Zariski-dense.

By Chevalley's theorem ([21, Theorem 3.16]) we also know that $\psi(\mathfrak{I}^P)$ is constructible; since every constructible Zariski-dense subset of an irreducible scheme contains a dense open set [21, Proof of Theorem 3.16], we have $X_b \cong F$ for all b in an open dense subset of B . This concludes the proof. \square

Proof of the Proposition in Section 1. By Lemma 4.10, all the fibres over a dense open subset $U \subset B$ are isomorphic, which shows (1). By [28, Proposition 2.6.10], there exists an étale cover $\epsilon: U' \rightarrow U$ such that the induced fibration $X'_{U'} := X \times_U U'$ is trivial:

$$X'_{U'} \cong U' \times F.$$

This shows (2).

Now suppose that the generic fibre F is of general type. This implies that the group $G := \text{Aut}(F)$ is finite; therefore, for any $x \in F$, we can define the subvariety

$$W^x := U' \times G \cdot x \subset X'_{U'}.$$

We are going to show that the image of W^x by the cover $\epsilon_X: X'_{U'} \rightarrow X_U$ is f -invariant.

Remark that the fibration $\pi_U: X_U \rightarrow U$ is locally trivial in the euclidean topology. Let $\{U_i\}_{i \in I}$ be a covering of U by euclidean open subsets such that the restriction of the fibration to each X_{U_i} is trivial: there exist biholomorphisms $X_{U_i} \cong U_i \times F$. Then the subvarieties

$$V_i^x := U_i \times G \cdot x \subset X_{U_i}$$

patch together to algebraic subvarieties V^x of X_U which are exactly the images of the W^x . Now we will prove that the varieties V^x are f -invariant. Let $p \in X_U$ be a point where f is defined and such that g is defined on $\pi(p)$, and let $i \in I$ be such that $p \in X_{U_i}$; up to shrinking U_i , we can suppose that $g(U_i) \subset U_j$. By an identification $X_{U_i} \cong U_i \times F$, $X_{U_j} \cong U_j \times F$, we can write $f(x, y) = (g(x), h(x, y))$; here, for all x in an open dense subset of U_i , the continuous map $b \mapsto h(x, \bullet) \in \text{Bir}(F)$ is well defined. Since $\text{Bir}(F)$ is a finite

(hence discrete) group, h doesn't depend on x , which shows that all the varieties V^x are f -invariant.

Now remark that the varieties $\epsilon_X(U' \times G \cdot \{x\})$ are the disjoint union of varieties of type $\epsilon_X(U' \times \{y\})$; since the first are f -invariant, the latter must be f -periodic, which concludes the proof. \square

Proof of the Main Theorem, point (1). Let X be an irreducible symplectic manifold, $f: X \dashrightarrow X$ a birational loxodromic transformation, and suppose by contradiction that f is imprimitive: there exist thus a meromorphic fibration $\pi: X \dashrightarrow B$ and a bimeromorphic transformation $g: B \dashrightarrow B$ such that $\pi \circ f = g \circ \pi$.

By Lemma 4.7, X is projective and the generic fibre of π is of general type. However, we also know by Lemma 4.6 that the generic orbit of f is Zariski-dense; therefore, by the Proposition in Section 1, the generic fibre of π cannot be of general type, a contradiction. \square

5. INVARIANT SUBVARIETIES

Let X be a compact complex manifold. If $f: X \rightarrow X$ is an automorphism, we say that a subvariety $W \subset X$ is *invariant* if $f(W) = W$, or, equivalently, if $f^{-1}(W) = W$. We say that $W \subset X$ is *periodic* if it is invariant for some positive iterate f^n of f .

Now let $f: X \dashrightarrow X$ be a pseudo-automorphism of X (i.e. a bimeromorphic transformation which is an isomorphism in codimension 1). We say that a hypersurface $W \subset X$ is invariant if the strict transform f^*W of W is equal to W (as a set); since f and f^{-1} don't contract any hypersurface, this is equivalent to $f(W) = W$ (here $f(W)$ denotes the analytic closure of $f|_U(W \cap U)$, where $U \subset X$ is the maximal open set where f is well defined). We say that a hypersurface is periodic if it is invariant for some positive iterate of f .

The following Theorem is a special case of [5][Theorem B].

Theorem 5.1. *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a compact complex manifold X . If f admits at least $\dim(X) + b_2(X) - 1$ invariant hypersurfaces, then it preserves a non-constant meromorphic function.*

Proof of the Main Theorem, point (2). Let $f: X \dashrightarrow X$ be a loxodromic bimeromorphic transformation of an irreducible symplectic manifold X (which is a pseudo-automorphism by 3.5).

Suppose that f admits more than $\dim(X) + b_2(X) - 2$ periodic hypersurfaces; then some iterate of f satisfies the hypothesis of Theorem 5.1. Therefore f^n preserves a non-constant meromorphic function $\pi: X \rightarrow \mathbb{P}^1$, and, up to considering the Stein factorization of π , we can assume that π is an f^n -invariant fibration onto a curve. This contradicts point (1) of the Main Theorem. \square

The following example shows that we cannot hope to obtain an analogue of point (2) of the Main Theorem for higher codimensional subvarieties.

Example 5.2. Let $f: S \rightarrow S$ be a loxodromic automorphism of a K3 surface S , and let $X = \text{Hilb}^n(S)$. Then X is an irreducible symplectic manifold and f induces a loxodromic automorphism f_n of X . By point (2) of the Main Theorem, f_n admits only a finite number of invariant hypersurfaces. However f admits infinitely many periodic points ([4],[6]); if x is a periodic point in S , then (the image in X of) $\{x\}^p \times S^{n-p}$ is a periodic subvariety of codimension $2p$.

Thus we have showed the following Proposition.

Proposition 5.3. *For all integers $0 < p \leq n$, there exist a $2n$ -dimensional projective irreducible symplectic manifold X and a loxodromic automorphism $f: X \rightarrow X$ admitting infinitely many periodic subvarieties of codimension $2p$.*

6. APPENDIX: AN ALTERNATIVE APPROACH TO THE MAIN THEOREM

In this section we describe a different approach to the proof of the Main Theorem which doesn't require the Proposition in Section 1. The result we obtain is actually slightly weaker than the Main Theorem; however this approach allows to prove point (2) and (3), as well as point (1) for automorphisms.

We have already seen in Proposition 3.7 that the first dynamical degree of a bimeromorphic transformation $f: X \dashrightarrow X$ is either 1 or an algebraic integer λ whose conjugates over \mathbb{Q} are λ^{-1} and some complex numbers of modulus 1 (so that λ is a quadratic or a Salem number). In the case of automorphisms, the following Proposition from Verbitsky [32] allows to completely describe all the other dynamical degrees as well.

Proposition 6.1. *Let X be an irreducible symplectic manifold of dimension $2n$ and let $SH^2(X, \mathbb{C}) \subset H^*(X, \mathbb{C})$ be the subalgebra generated by $H^2(X, \mathbb{C})$. Then we have an isomorphism*

$$SH^2(X, \mathbb{C}) = \text{Sym}^* H^2(X, \mathbb{C}) / \langle \alpha^{n+1} | q_{BB}(\alpha) = 0 \rangle$$

The following Corollary is due to Oguiso [27].

Corollary 6.2. *Let $f: X \rightarrow X$ be an automorphism of an irreducible symplectic manifold of dimension $2n$. Then for $p = 0, 1, \dots, n$*

$$\lambda_p(f) = \lambda_{2n-p}(f) = \lambda_1(f)^p.$$

Proof. By Proposition 6.1 the cup-product induces an injection

$$\text{Sym}^p H^2(X, \mathbb{C}) \hookrightarrow H^{2p}(X, \mathbb{C})$$

for $p = 1, \dots, n$.

Let $v_1 \in H^2(X, \mathbb{C})$ be an eigenvector for the eigenvalue $\lambda = \lambda_1(f)$. Then $v_p := v_1^p \in H^{2p}(X, \mathbb{C})$ is a non-zero class for $p = 1, \dots, n$ and $f^*v_p = (f^*v_1)^p = \lambda^p v_p$. This implies that $\lambda_p(f) \geq \lambda_1(f)^p$, and we must have equality by log-concavity (Proposition 2.8). This proves the result for $p = 0, 1, \dots, n$.

Now by Remark 2.3 we have $\lambda_{2n-p}(f) = \lambda_p(f^{-1})$. Applying what we have just proved to f^{-1} we obtain

$$\lambda_{2n-1}(f) = \lambda_1(f^{-1}) = \lambda_n(f^{-1})^{1/n} = \lambda_n(f^{-1})^{1/n} = \lambda_1(f)$$

and thus, for $p = 0, \dots, n$,

$$\lambda_{2n-p}(f) = \lambda_p(f^{-1}) = \lambda_1(f^{-1})^p = \lambda_1(f)^p,$$

which concludes the proof. \square

Lemma 6.3. *Let X be a smooth projective variety, $f: X \dashrightarrow X$ a birational transformation of X , $\pi: X \dashrightarrow B$ a rational f -invariant fibration onto a smooth projective variety B . If the generic fibre of π is of general type, then all the relative dynamical degrees $\lambda_p(f|\pi)$ are equal to 1 (for $p = 0, \dots, \dim(X) - \dim(B)$).*

Proof. Since the Kodaira dimension and the relative dynamical degrees are bimeromorphic invariants (Remark 2.5), up to considering a resolution of the indeterminacy locus of π , we can suppose that π is regular.

Let

$$\begin{array}{ccc} \phi: X & \longrightarrow & Y := \text{Proj}(\pi_* K_{X/Y}^{\otimes m}) \\ \downarrow \pi & \nearrow \eta & \\ B & & \end{array}$$

be the Iitaka fibration. Since ϕ is birational onto its image, denoting $Z \subset Y$ the closure of $\phi(X)$, the claim is equivalent to $\lambda_p(f_Z|\eta_Z) = 1$, where η_Z denotes the restriction of η to Z and $f_Z = \phi \circ f \circ \phi^{-1}: Z \dashrightarrow Z$.

The construction of Remark 4.9 provides a birational transformation $\tilde{f}: Y \dashrightarrow Y$ extending f_Z .

$$\begin{array}{ccccc} \begin{array}{c} \curvearrowright f \curvearrowright \\ X \end{array} & \xrightarrow{\phi} & \begin{array}{c} \curvearrowright f_Z \curvearrowright \\ Z \end{array} & \xrightarrow{\quad} & \begin{array}{c} \curvearrowright \tilde{f} \curvearrowright \\ Y \end{array} \\ \downarrow \pi & & \nearrow \eta & & \\ \begin{array}{c} \curvearrowright \curvearrowright \\ B \end{array} & & & & \end{array}$$

Now we will prove that if $\lambda_p(\tilde{f}|\eta) = 1$ then $\lambda_p(f_Z|\eta_Z) = 1$. Let $H_Y \in \text{Pic}(Y)$ and $H_B \in \text{Pic}(B)$ be ample classes; therefore $H_Y|_Z$ is an ample class on Z . The map

$$\begin{aligned} H^{2n,2n}(Y, \mathbb{R}) &\rightarrow \mathbb{R} \\ \alpha &\mapsto \int_Y \alpha \wedge c_1(H_Y)^{\dim(Y)-\dim(X)} = \alpha \cdot H_Y^{\dim(Y)-\dim(X)} \end{aligned}$$

is linear and strictly positive (except on 0) on the closed positive cone $\mathcal{K}_{2n} \subset H^{2n,2n}(Y, \mathbb{R})$. Since $\alpha \mapsto \alpha \cdot [Z]$ is linear too, we can define

$$M = \max_{\alpha \in \mathcal{K}_{2n} \setminus \{0\}} \frac{\alpha \cdot [Z]}{\alpha \cdot H_Y^{\dim(Y)-\dim(X)}} \geq 0.$$

Now

$$\begin{aligned} \lambda_p(f_Z|\eta_Z) &= \lim_{n \rightarrow +\infty} \left((\tilde{f}^n)^* H_Y^p \cdot \eta^* H_B^{\dim(B)} \cdot H_Y^{2n-p-\dim(B)} \cdot [Z] \right)^{\frac{1}{n}} \leq \\ &\lim_{n \rightarrow +\infty} \left(M(\tilde{f}^n)^* H_Y^p \cdot \eta^* H_B^{\dim(B)} \cdot H_Y^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}} = \\ &\lim_{n \rightarrow +\infty} \left((\tilde{f}^n)^* H_Y^p \cdot \eta^* H_B^{\dim(B)} \cdot H_Y^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}} = \lambda_p(\tilde{f}|\eta) = 1, \end{aligned}$$

and since all relative dynamical degrees are ≥ 1 (Proposition 2.8) we have $\lambda_p(f_Z|\eta_Z) = 1$.

Now all is left to prove is that $\lambda_p(\tilde{f}|\eta) = 1$. There exists $k > 0$ such that $\eta^* H_B^{\dim(B)} \equiv_{\text{num}} k[F]$, where $[F]$ is the numerical class of a fibre F of η . We have

$$\begin{aligned} \lambda_p(\tilde{f}|\eta) &= \lim_{n \rightarrow +\infty} \left((\tilde{f}^n)^* H_Y^p \cdot \eta^* H_B^{\dim(B)} \cdot H_Y^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}} = \\ &\lim_{n \rightarrow +\infty} \left((\tilde{f}^n)^* H_Y^p \cdot k[F] \cdot H_Y^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}} = \\ &\lim_{n \rightarrow +\infty} \left(\left((\tilde{f}^n)^* H_Y \right) |_F^p \cdot H_Y|_F^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}}. \end{aligned}$$

For each fibre we have a canonical identification $F \cong \mathbb{P}^N$, and by this identification $H_Y|_F \cong \mathcal{O}_{\mathbb{P}^N}(d)$, meaning that the hyperplane section $H_Y|_F$ is defined by an equation of degree d . Under the identification, the action of \tilde{f} from one fibre to another is linear, so that $((\tilde{f}^n)^* H_Y)|_F$ is also defined by an equation of degree d on \mathbb{P}^N . This means that

$$\lambda_p(\tilde{f}|\eta) = \lim_{n \rightarrow +\infty} (d^{\dim(F)})^{\frac{1}{n}} = 1$$

as we wanted to show. This concludes the proof. \square

The following Proposition is a weaker version of point (1) of the Main Theorem.

Proposition 6.4. *Let $f: X \dashrightarrow X$ be a loxodromic transformation of an irreducible symplectic manifold X of dimension $2n$, and let*

$$1 = \lambda_0(f) < \cdots < \lambda_{p_0}(f) = \cdots = \lambda_{p_0+k}(f) > \cdots > \lambda_{2n}(f) = 1$$

be its dynamical degrees.

If $\pi: B \dashrightarrow B$ is an f -invariant meromorphic fibration, then $\dim(B) \geq 2n - k$. In particular, if f is an automorphism (or, more generally, if all the consecutive dynamical degrees of f are distinct), then it is primitive.

Proof. Let $g: B \dashrightarrow B$ be a birational transformation such that $g \circ \pi = \pi \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ | & & | \\ \pi & & \pi \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & B \end{array}$$

We know by Lemma 4.7 that the generic fibre of π is of general type; by Lemma 6.3 this implies that all the relative dynamical degrees $\lambda_p(f|\pi)$ are equal to 1. By Theorem 2.7 we then have

$$\lambda_p(f) = \max_{p - \dim(F) \leq q \leq p} \lambda_q(g),$$

where $\dim(F) = \dim(X) - \dim(B)$ is the dimension of a generic fibre.

Let $q \in \{0, 1, \dots, \dim(B)\}$ be such that $\lambda_q(g)$ is maximal. Then

$$\lambda_q(f) = \lambda_{q+1}(f) = \cdots = \lambda_{q+\dim(F)}(f) = \lambda_q(g),$$

meaning that $k \geq \dim(F) = 2n - \dim(B)$. This concludes the proof. \square

Remark 6.5. Since in the Theorem we have $k \leq 2n - 1$, the base of an invariant fibration cannot be a curve. Therefore Proposition 6.4 implies point (2) of the Main Theorem.

REFERENCES

- [1] Ekaterina Amerik and Frédéric Campana. Fibrations méromorphes sur certaines variétés à fibré canonique trivial. *Pure Appl. Math. Q.*, 4(2, part 1):509–545, 2008.
- [2] Garrett Birkhoff. Linear transformations with invariant cones. *Amer. Math. Monthly*, 74:274–276, 1967.
- [3] Sébastien Boucksom. Divisorial Zariski decompositions on compact complex manifolds. *Ann. Sci. École Norm. Sup. (4)*, 37(1):45–76, 2004.
- [4] Serge Cantat. Dynamique des automorphismes des surfaces $K3$. *Acta Math.*, 187(1):1–57, 2001.
- [5] Serge Cantat. Invariant hypersurfaces in holomorphic dynamics. *Math. Res. Lett.*, 17(5):833–841, 2010.
- [6] Serge Cantat. Dynamics of automorphisms of compact complex surfaces. In *Frontiers in complex dynamics*, volume 51 of *Princeton Math. Ser.*, pages 463–514. Princeton Univ. Press, Princeton, NJ, 2014.

- [7] Serge Cantat, Antoine Chambert-Loir, and Vincent Guedj. *Quelques aspects des systèmes dynamiques polynomiaux*, volume 30 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2010.
- [8] Jean-Pierre Demailly. *Complex analytic and differential geometry*. Citeseer, 1997.
- [9] J. Diller and C. Favre. Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.*, 123(6):1135–1169, 2001.
- [10] Tien-Cuong Dinh and Viêt-Anh Nguyễn. Comparison of dynamical degrees for semi-conjugate meromorphic maps. *Comment. Math. Helv.*, 86(4):817–840, 2011.
- [11] Tien-Cuong Dinh, Viêt-Anh Nguyễn, and Tuyen Trung Truong. On the dynamical degrees of meromorphic maps preserving a fibration. *Commun. Contemp. Math.*, 14(6):1250042, 18, 2012.
- [12] Tien-Cuong Dinh and Nessim Sibony. Regularization of currents and entropy. *Ann. Sci. École Norm. Sup. (4)*, 37(6):959–971, 2004.
- [13] Tien-Cuong Dinh and Nessim Sibony. Upper bound for the topological entropy of a meromorphic correspondence. *Israel J. Math.*, 163:29–44, 2008.
- [14] Marat Kharisovitch Gizatullin. Rational g -surfaces. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 44(1):110–144, 1980.
- [15] Julien Grivaux. Parabolic automorphisms of projective surfaces (after mh gizatullin). *arXiv preprint arXiv:1307.1771*, 2013.
- [16] M. Gromov. Convex sets and Kähler manifolds. In *Advances in differential geometry and topology*, pages 1–38. World Sci. Publ., Teaneck, NJ, 1990.
- [17] M. Gross, D. Huybrechts, and D. Joyce. *Calabi-Yau manifolds and related geometries*. Universitext. Springer-Verlag, Berlin, 2003. Lectures from the Summer School held in Nordfjordeid, June 2001.
- [18] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.
- [19] A. Grothendieck and J. A. Dieudonné. *Éléments de géométrie algébrique. I*, volume 166 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1971.
- [20] Vincent Guedj. Decay of volumes under iteration of meromorphic mappings. *Ann. Inst. Fourier (Grenoble)*, 54(7):2369–2386 (2005), 2004.
- [21] Joe Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course, Corrected reprint of the 1992 original.
- [22] Fei Hu, JongHae Keum, and De-Qi Zhang. Criteria for the existence of equivariant fibrations on algebraic surfaces and hyperkähler manifolds and equality of automorphisms up to powers: a dynamical viewpoint. *J. Lond. Math. Soc. (2)*, 92(3):724–735, 2015.
- [23] AG Khovanskii. The geometry of convex polyhedra and algebraic geometry. *Uspehi Mat. Nauk*, 34(4):160–161, 1979.
- [24] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.
- [25] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 257–322. Springer, Heidelberg, 2011.
- [26] BG Moishezon. A criterion for projectivity of complete algebraic abstract varieties. *Amer. Math. Soc. Transactions*, 63:1–50, 1967.
- [27] Keiji Oguiso. A remark on dynamical degrees of automorphisms of hyperkähler manifolds. *Manuscripta Math.*, 130(1):101–111, 2009.
- [28] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
- [29] Bernard Teissier. Du théorème de l’index de Hodge aux inégalités isopérimétriques. *C. R. Acad. Sci. Paris Sér. A-B*, 288(4):A287–A289, 1979.
- [30] Hajime Tsuji. Global generation of the direct images of relative pluricanonical systems. *arXiv preprint arXiv:1012.0884*, 2010.
- [31] Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces*. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975. Notes written in collaboration with P. Cherenack.
- [32] M. Verbitsky. Cohomology of compact hyper-Kähler manifolds and its applications. *Geom. Funct. Anal.*, 6(4):601–611, 1996.